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AN

ESSAY

For the Discovery of Some NEW

Geometrical Problems,

(Judged by some Learned Men, Impracticable)
Concerning

ANGULAR SECTIONS,

Beginning with the

GEOMETRICAL TRISECTION of any Right Lined Angle, by Plain *Geometry* of Right Lines and Arches of Circles,

With *RULE* and *COMPASS* only, without all Conick Sections, and Cubick Æquations.

Whether the following Praxis, and apparent demonstration thereof doth not only make it Practicable, but easie to the Understanding of a Tiro, who but understands a little in true Geometrical Learning.

Which layeth a Foundation of a Plain Method how to Sect any Angle into any other Number of Parts required, *Even* as 4. 6. 8. 10; or *Uneven*, as 5. 7. 9. 11. *&c.* As also to divide a Circle into any number *Even*, or *Uneven* of equal parts.

All which have great Uses in the Improvement of the Mathematical Sciences, some of which are here specified.

Proposed and Submitted to the Impartial Tryal and Examination of the Right Reason of such Artists, to whose Hands it may come.

By G. K.

London, Printed 1697. And to be Sold by the Author, at his House in *Pudding-Lane*, at the Sign of he *Golden-Ball*, near the *Monument*; And by *B. Aylmer* at the *Three Pigeons* over against the *Royal-Exchange*.

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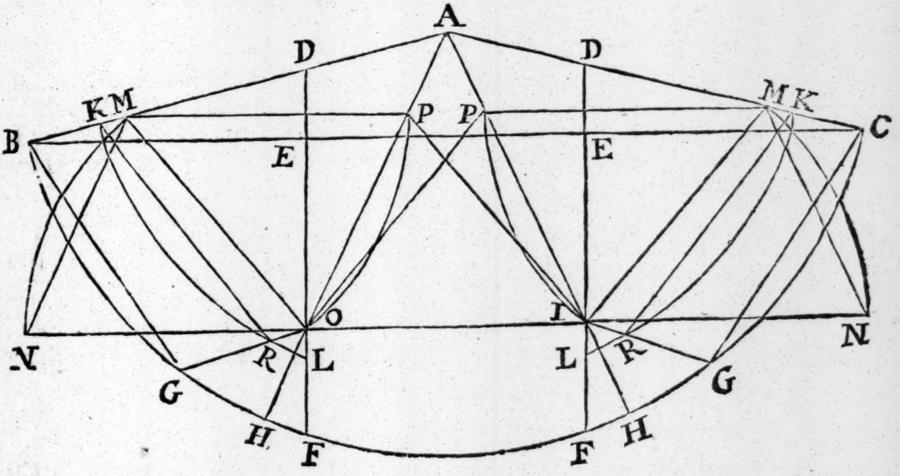
Some New Geometrical Problems, &c.

ALthough the Trisection of a right lined Angle and also its Section into any parts required, to a true Mathematical exactness, is denyed to be practicable by plain *Geometry* (of right Lines and Arches of Circles, with Rule and Compass only, without all Conick Sections, and Algebra Equations) by some learned Artists; yet others as learned are not so positive, but acknowledge the Practise of it is not as yet discovered, among whom is the learned *Is. Barrow*, who hath writ thus, in Corol. ad. 9. 1. elem. Eucl.

Methodus vero regula & Circino angulos secandi in æquales quotcunq; hactenus Geometras latuit. and in Schol. ad 16. 4. Elem. *Cæterum divisio circumferentiæ in partes datas etiamnum desideratur.*

The Praxis of the Trisection of any given Angle.

Suppose the given Angle to be the Angle of 150 Degrees, made by the addition of 90 d. the Quadrant, and 60 d. the $\frac{1}{6}$ of the Circle, whose exact Mathematical quantity is known. And let that Angle be in the subsequent Figure, the Angle BAC. whose Cord is BC, and the Arch answering to that Cord BFC.



1. Divide

1. Divide the Cord BC into 3 equal Parts, as BE=EE=EC by 10. 6. elem. Euclid. and draw the 2 Perpendiculars EF. EF.

2. With the extent of $\frac{1}{3}$ of the Cord as EC measure on the Arch from C to G, and draw the Line or Cord GC= EC.

3. With another Radius less than AC, *viz.* AK draw a second arch as KL, until it cut the perpendicular EF, and let the Radius AK be so long, that the extent of the Cord betwixt K and L be somewhat longer than EC, and with the same extent EC measure on this second arch KR=EC.

4. From G to R draw a straight Line by post 1. 1. *elem.* and produce the same by post 2. until it cut the Perpendiculars at I and O.

5. Draw the straight lines AI and AO, and extend them to H on both sides, so that the one line shall be AIH, and the other AOH, which two Lines shall sect the given angle BAC into three parts or angles, *viz.* BAH, HAH, HAC.

The Question, or Problem to be resolved is, whether these three angles are not equal, and consequently that the angle BAC is trisected into equal angles.

The apparent demonstration that is here proposed to prove it, followeth.

In order to the apparent following demonstration, by way of Preparation,

1. Draw the line OI, which shall be parallel to EE by 28. 1. *elem.* and extend it on both sides, from I to N on the one side, and from O to N on the other, making IN=ON=EC=OI.

2. With Radius IN, and ON, describe the arches on both sides NM until these arches cut the lines AC and AB, so shall the lines or cords drawn betwixt I and M and O and M, *viz.* IM and OM =EC.

3. Make the angle IMP=angle MIN by 23. 1. *elem.* and OMP=MON, and extend the line MP, until it cut the line AI at P, and AO at P.

4. Draw the line OP parallel to IM, until it cut MP, whence the perfect Rhombus OMPI shall be formed, having the opposit sides paralel and equal, for PM by construction is parallel with OI, and OP with IM and that the Line OP drawn parallel to IM can cut the Line PM no where but at P on the line AI is proved thus. Suppose it cut the line PM any where else, than on the line AI on either side, it should make PM either longer or shorter than its opposit side (for the arch drawn by the radius MI, and cutting the line AI at P proveth MP=MI by *def.* 15. 1. *elem.*) as also OP should be longer or shorter than IM, (as the like arch

as

conceived to be drawn by radius OI, and cutting AI at P, proveth OI=OP by the ſame, *def.* 15. 1. *elem.*) but both theſe Conſequences are abſurd, making the oppoſit ſides of a Paraleliogram to be unequal, contrary to 34. 1. *elem.*

Becauſe the Figure OPMI is proved to be a perfect Rhombus, having all the ſides equal, whereof the line PI is the diagonal, (it being proved that OP and PM terminate on AI) therefore the angle OIP (=OIA) = PIM =(AIM) by 8. and 4. 1. *elem.*

Laſtly, theſe two Triangles OAI and IAM ſhall be according to the 4th. prop. 1. *elem.* and ſo ſhall OAI and OAM, for OI=IM as above proved, and AI is common to both, and the Angle OIA=AIM as is above proved, therefore by 4. 1. *elem.* AM=AO =AI, and conſequently by 8. 1. the angles IAM=OAI=OAM, therefore the given angle BAC is Geometrically triſected by the Lines AO and AI extended to H. Q. E. D.

Here *Note*, let the Radius be ever ſo much changed betwixt M and C, taking it at any extent from A, and let ever ſo many Concentrick Arches be drawn from the Center A betwixt IM and GC, their Cords terminating on the ſtreight Lines MC on the one ſide, and IG on the other ſhall all be equal one to another, and to the Cord GC=⅓ BC. as the demonſtration above given proveth, for the reaſon that proveth the Cords of any 3 Concentrick Arches, terminating on two ſtreight Lines to be equal, within the limits MI and GC, being a Trapezia, proveth any other 3 to be equal within the ſame limits. But if we draw any Concentrick Arches without the Trapezia IGCM, as with a leſs Radius, than AM, or with a greater Radius, than AC, the Caſe is altered, and the equal Cords will not terminate on the ſtreight Line IG, but deviate or depart therefrom, as both true Reaſon doth prove, and even ocular inſpection doth ſhew; for though the Eye is not able to judge of a ſtraight Line, yet when a Line is viſibly apparent to make an Angle with another Line, theſe two Lines cannot be a ſtraight Line. I call the Figure IMCG a Trapezia, for it is not any Paralellogram, becauſe the ſide IG, is not parallel to the ſide MC, but very unparalel, which makes the ſide IG to be longer than the ſide MC, but this doth not hinder that all the Cords of the Concentrick Arches drawn betwixt IG and MC are equal, for let them be ever ſo many within the limits IM and GC, they are all equal, though their Arches are unequal being from a differing Radius. And though the demonſtration above given may ſeem ſufficient to prove it, yet for a further proof, let a Line be drawn, or conceived to be drawn (as in this Figure it is only conceived) from I Paralel to AC, towards

FC

FC and beyond it, all the extream Points of the Cords of the Concentick Arches, above mentioned, terminating on the ſtreight line IG, do gradually ſtill more and more depart from that parallel towards the perpendicular IF within the aforeſaid limits, IM and GC, on the firſt Arch, which I call the primitive Arch; but if we draw any Arch, without GC by a longer Radius than AC, or within IM by a ſhorter Radius than AM, the Cords of thoſe concentrick Arches, if equal, ſhall deviate from the ſtreight line IG, and come nearer to the Parallel above mentioned, but that within the Trapezia above ſaid IMCG, the equal Cords of all theſe Arches ſhall terminate on the ſtraight line IG I thus further prove. Draw, (or conceive as drawn in this Figure) a ſtraight Line from C to H parallel to IM, and take a Cord of any of thoſe Concentrick Arches $= \frac{1}{3}$ of the Cord BC=IM, I ſay it muſt terminate on the line IG, and neither go without it, nor within it, otherwiſe the ſame extent cutting the paralel Lines, *viz.* the one paralel to MC, the other to IM, ſhall make the oppoſite ſides of a Paralellogram unequal contrary to 34. 1. *elem. Eucl.* as the due Conſideration of the Figure will ſhow, and the part equal to the whole.

I have in the Example of Triſection uſed an obtuſe Angle, becauſe it is more commodious and eaſie to be done by Manual Operation in an obtuſe Angle than in an Acute as Experience will ſhow, though the demonſtration is the ſame both in the Obtuſe and Acute. Nor is it any juſt exception, that ſome acute Angles being ſmall, can hardly be triſected by this Method; for when Angles are very acute, they can hardly be biſected. But the proper Remedy for both is, when the Angle is ſmall, double or quadruple it, and then biſect or triſect it, as occaſion requires.

The Praxis of the Triſection hath theſe following great Uſes.

By the Triſection of the Angle of 120 degrees we have the true Mathematical Cord of 40 degrees, and ſeeing it is demonſtrated in 11. 4. *elem. Euclid.* to find the Cord of $\frac{1}{5}$ of the Circle, =72 degrees the $\frac{1}{2} = 36$ by ſubſtracting 36d. from 40d. we have the true Cord of 4 dergees, that biſected gives the Cord of 2 degrees, and that again biſected gives the Cord of 1 degree. Alſo 40 degrees twice biſected gives 10 degrees, and that again biſected gives 5. and thus by the Triſection of 120 or 60 degrees, and certain other biſections a line of Cords truly Geometrical or Mathematical can be made by plain Geometry, without all conick Sections or Algebra Æquations, and tedious Extractions of Roots, and alſo without any Table of natural Sines, which never hitherto hath been taught (ſo far as I ever heard or read) in a plain method, by plain Geometry, ſo as to be

be made intelligible to any Tiro or Young Artiſt; and yet ſuch a method is altogether neceſſary for the perfection of Geometry and Mathematical Learning, that the way to triſect any Angle be known by plain Geometry, ſeeing many things in plain Geometry require a true line of Cords, and a true line of Cords cannot be found without the Triſection of certain Angles, or ſome other Section than Biſection; and to refer a Tiro, or Young Scholar, to the conick Sections, and tedious Algebra Æquations and Reſolutions how to triſect an Angle, or underſtand how to make a true Geometrical Line of Cords is altogether immethodical, even as much or more as to referr him to ſome of the difficulteſt Problems in *Euclid's Elements* to underſtand the demonſtration of one of the firſt Propoſitions in thoſe Elements. Seeing therefore *Geometry* as it is one of the beſt natural Sciences for certainty and uſe, ſo for good Method; and the nature of good Method requireth that in teaching we proceed to the *more unknown* from and by the *more known*, therefore to preſuppoſe the knowledge of *conick Sections* to the knowledge of ſome neceſſary Problems in plain Geometry, is greatly incongruous and an immethodical Hyſteron Proteron, which this new Praxis doth Remedy.

2. By this Praxis of Triſection a Foundation of Method is laid to ſect any Angle given into any other equal parts whatſoever, *even* as 4. 6. 8. 10. or *uneven* as 5. 7. 9. 11. *&c.* and an univerſal Cannon is formed, as by triſection of the Cord of an Angle, that Angle is triſected, ſo by quinquiſection of the Cord of an Angle it is quinquiſected, and the like of all others.

3. By the like method a Circle is divided into any number of equal parts even or uneven.

4. And, by the like Method, the quantity of any Right-lined Angle can be found in degrees, and odd Minutes, without any line of Cords by a line of equal parts only, as the diagonal Scale of Inch, or half Inch or $\frac{1}{4}$ to a far greater exactneſs than by a line of Cords.

5. By the ſame, a way is taught how to protract or project any Angle, whoſe degrees and odd Minutes are given.

6. Having any one Angle given in any Triangle, and the Ratio of the 2 other Angles, without any ſide given, to find the other 2 Angles, and truly to protract them.

7. From a point given on any line given to raiſe an Iſoſceles Triangle, one of whoſe ſides being produced below that line given ſhall terminate on any Point, given below the ſame, the which hath ſpecial uſe to ſolve ſome new Problems in Surveying, Geography, Architecture, Navigation.

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8. As this method of Praxis saveth the pains of finding out the Trisection of an Angle by a conick Section, and Algebra Æquations, by teaching the same more easily by plain Geometry, so it is probable it may prove of great use, both in conick Sections, Algebra Questions, and other abstruse Theorems to improve the Mathematical Sciences.

9. How to find the natural Cord, Sine, Tangent, Secant of any Angle given in degrees and Minutes, in true Lines, more exactly, than by any common Line of Cords.

Here followeth another Demonstration of the same Praxis of the Trisection.
In a Second Figure.

BY the method of the foregoing Praxis, let the points I and O be found, and the lines AI and AO, which I say shall trisect the Angle NAK=BAC, Geometically into three equal parts.

In order to the demonstration of which, extend the line AI to D on the arch BFC. and AO to X.

Again, with the extent AI on the center I describe the circle ASHT. and with the same Radius describe the arch KLON.

Again, draw IH parallel to AK, extending IH to the Circumference of the Circle ASHT.

Again, draw AS parallel and equal to OI, which shall cut the circle at S, because OI=AS cuts the arch KION, having the same Radius.

Again, from the point S on the circumference, draw the line SH parallel to AI, which shall necessarily cut the line IH on the circumference at the point H, because by construction IH is parallel to AK, and KH to AI, which makes the Parallogram AKHI, whose opposit sides are equal, but if SH, and IH did meet any where else than on the circumference, as either within, or without it, the opposit sides of the Parallogram should not be equal contrary to 34. 1. *Eucl. elem.*

Again, draw the line SI, and extend the line OI from I to K until it cut SH at K, which shall form another Parallogram ASKI, which being divided by the Diagonal line SI, shall give two equal Isosceles Triangles AIS (=OAI) ISK.

The Demonstration.

In the Isoceles triangle ISH (whose sides IS=IH by 15.*def.* 1.*elem.*) the angle IHS=ISH by 5. 1. *elem.* and IHS=IAK by 8. 1. *elem.* therefore IAK=ISH. but ISH=AIS=OAI being alternate Angles betwixt Parallels by 29. 1. *elem.* and therefore lasty, OAI=IAK, and by the like method OAI is proved=OAN drawing the like parallel Lines on the other side of the Triangle OAI, and therefore the angle BAC is trisected by 3 equal angles NAO=OAI=IAK=BAX=XAD=DAC. Q. E. D.

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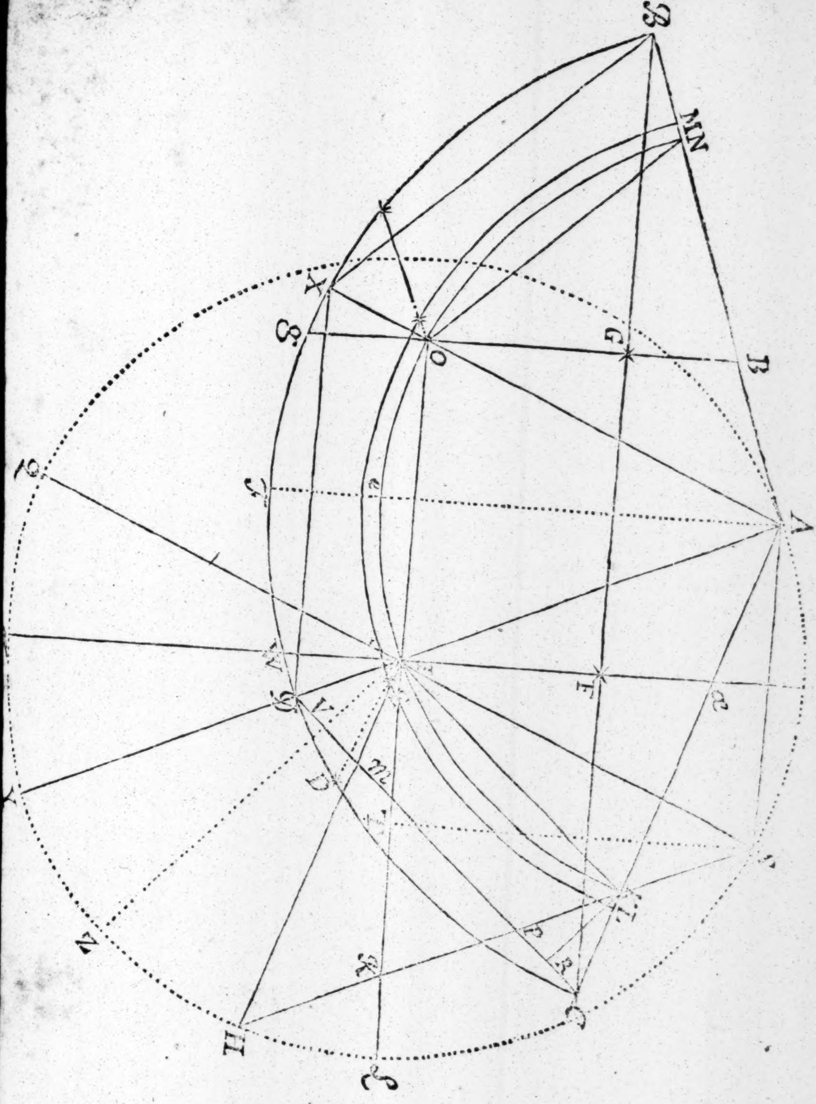
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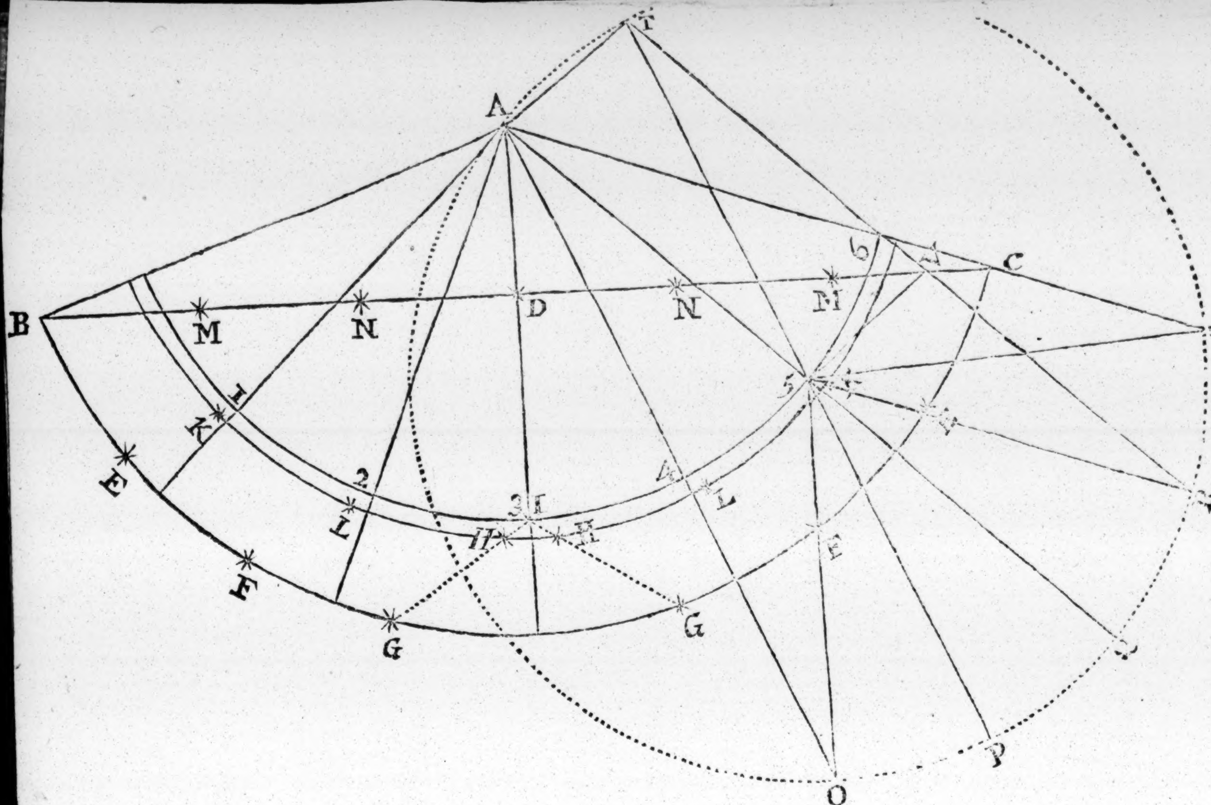

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The Praxis of the Section of a Right Lined Angle into Six equal parts.

BEcause of the affinity of the Figure of the Trisection with the Figure of the Sextisection, I shall begin with the Sextisection, and then proceed to the Quinquisection.

In the 3d figure, let the given angle be BAC, as above, whose Cord is BC, and its arch BEFGC.

1. Divide the Cord BC into 6 equal parts, which Divisions are marked with *****. and on the middle division draw the line ADI to the arch BEFGC.

2. With the extent of one of the Divisions, on the given arch measure from C to F, and with the same extent from F to E, and from E. to G, marking the points E. F. G. and with the same extent measure from B to E, from E to F, and from F to G marking the points E F G.

3. With a less Radius than AC as AK draw a second Arch as KLHLK. which Radius must be so long that the extent of the 1 6th of the Cord thrice repeated, may fall short of reaching the line ADI.

4. With the same extent on this 2d. arch, measure from AC, and from AB to K, and from K to L, and from L to H, by a three-fold repitition of the same extent.

5. From the point G on the first arch to the point H on the second arch, draw a straight line, and produce it until it cut the line ADI at the point I.

6. With the Radius AI draw a 3d. concentrick Arch, betwixt the lines AB and AC, I say the extent of the 1 6th part of the cord BC= MC, being 6 times repeated shall cut the said 3d arch into six equal parts to every one of which straight lines drawn from the Center A, and produced to the outmost arch, shall divide the given angle BAC into six equal parts orangles, marked with 1. 2. 3. 4. 5. 6.

In order to the demonstration of the Praxis of this Section of an angle into 6 equal parts, on the point 5, making it a center with Radius 5A describe the circle A O P Q R S.

2. From the point 5 draw the line 5 R parallel to A C.

3. From R draw the line RT parallel to 5A

4. Draw the line AT from A to T

5. Draw the line T 5, and draw 5 V parallel to AT

The Demonstration.

In the triangle T 5 R being an Isosceles, the angle 5 TR=5 R T. by 5. 1. *elem.* and 5 R T =(5 R 6) 5 A 6 therefore 5 T 6= 5A6.

Again 5 T 6=(5 T V) A 5 T, and A5 T = 5 A4 by 34. 1. and therefore lastly 5 A 4=5 A V. and as these 2 angles are proved equal by the like reasoning, all the other angles can be proved equal by changing the center of the Circle from 5 to 4, and from 4 to 3, and from 3 to 2, and from 2 to 1. This demonstration having such affinity with the former of Trisection, and which can as easily be demonstrated both ways, as that of Trisection, I shall not enlarge upon it, for he who understands and assents to the verity of the former, by the same evidence will assent to the later.

The Praxis of the Quinquisection.

1. ACcording to the former method, divide the cord of the given angle, BE into 5=parts.

2. On the middle part marked with CD, draw the lines CH and DH cutting the cord at right angles and parallel one to another.

3. On the arch of the given angle BFG take the $\frac{1}{5}$ of the cord = CD and set it from E to F, and from F to G, also do the like from B to F, and from F to G.

4. Draw a second arch, with a less radius as AK, so as the radius may be so long that the extent of the $\frac{1}{5}$ of the cord twice taken on that second arch may not reach to the perpendicular DH.

5. With the same extent twice taken, as from K to * from * to * measure on the second arch to the second *.

6. From G to * draw a straight line, until it cut the perpendicular line DH at H; lastly with the radius AH describe the third arch IHHI, and draw the lines AH AH which shall make the angle HAH=$\frac{1}{5}$ BAE.

The Demonstration of this, or any other Section, being after the same method with the former, as also the Praxis, it were superfluous to enlarge upon it, or to add any new Problems, showing how to sect any angle into any other parts given, as 7. 8. 9. 10. 11. *&c.* for he who understands by the foregoing Method and Praxis to sect any angle into 3. 5. 6. as is above shewed, will by the like Method and Praxis be able to sect any angle into 7. 8. 9. 10. *&c.* equal parts, and also to demonstrate the same.

Wherefore in the next place I shall proceed to shew how a Semicircle may be sected into any number of equal parts, *even* as 4. 6. 8. 10, *&c.* or *uneven*, as 5. 7. 9. 11.

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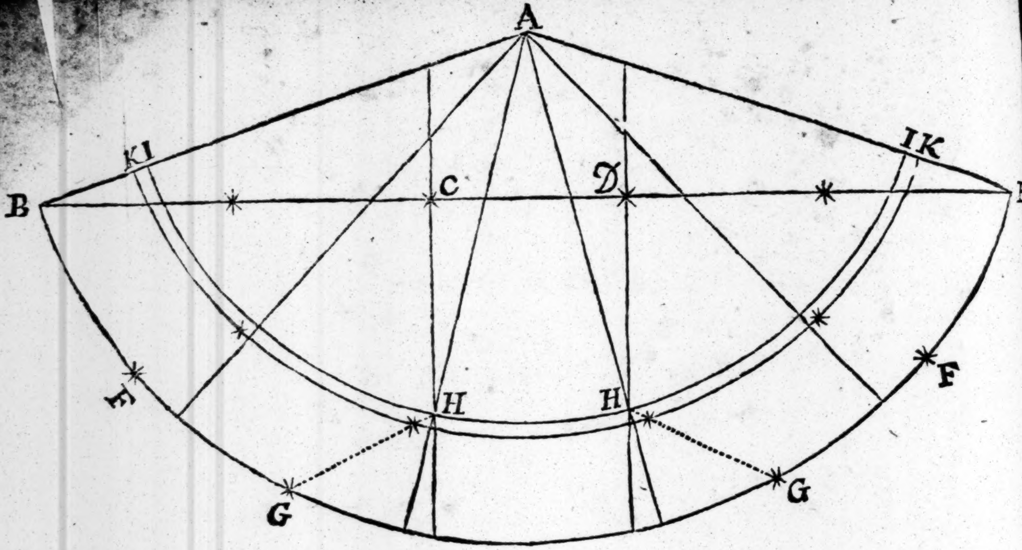
and affording new Problems in *Geography* and *Navigation*, which I
shall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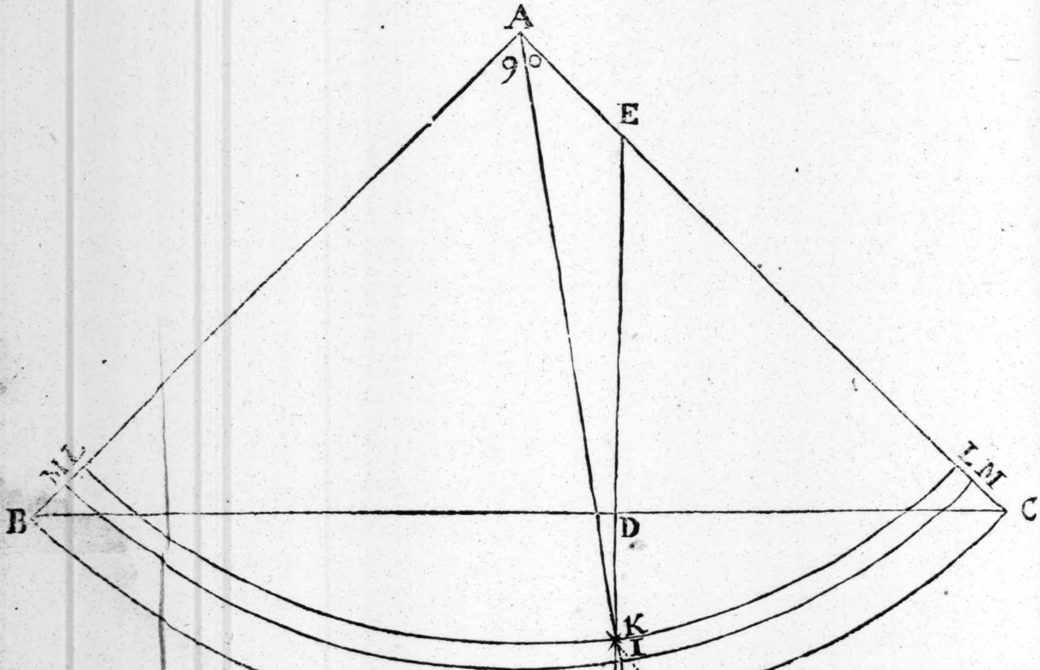
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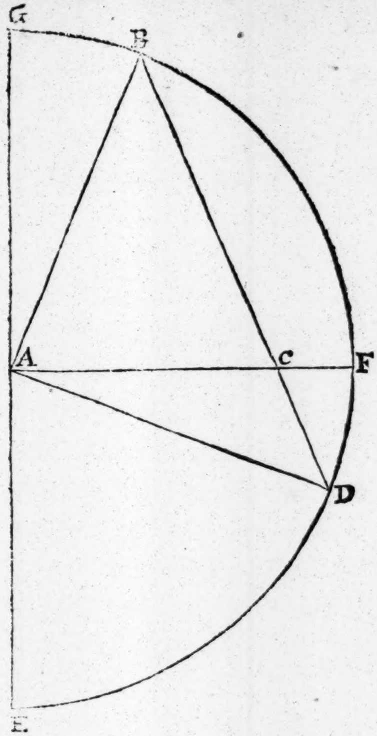
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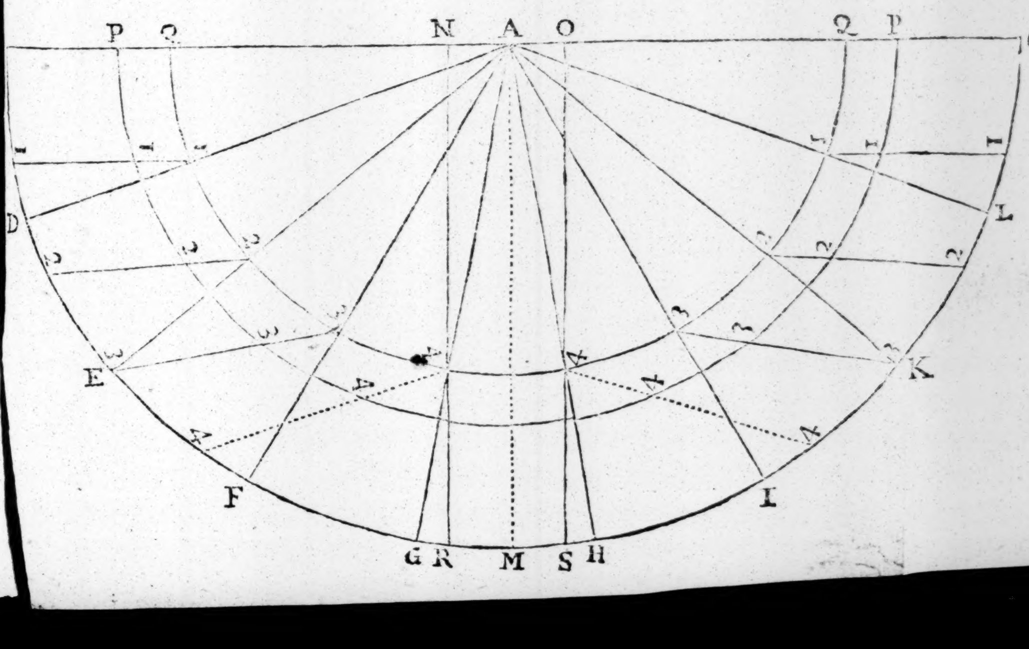
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The Praxis of a Section of a Semicircle into 9 equal parts.

Let the Semicircle given be BRSC, whose diameter is BC. and whose center is A.

1. Divide the diameter BC into 9 equal parts.

2. From the center A set off AO and AN, making NO=$\frac{1}{9}$ of the diameter: and draw the perpendiculars NR. OS.

3. With the extent NO measure from C to I, from 1 to 2, from 2 to three, from 3 to 4, marking the points 1. 2. 3. 4. and do the like from B to 1. 2. 3. 4.

4. With a less radius as AP describe the Semicircle PP, and from P to 1 with the same extent on that 2d. arch measure from P, P 1= 1. 2=2. 3=3. 4. and do the like from P on the other side. This lesser Radius must be so long, that the extent =NO four times repeated do not reach the Perpendiculars.

5. From 4 to 4 draw a straight line, and produce it until it cut OS at 4. and NR at 9.

6. Draw the 3d arch QQ with radius A 4, and draw the straight lines 4 A 4, which shall make an angle = a 9th part of the semicircle. The demonstration whereof is apparent from these precedent.

From the above delivered Praxis of the various Sections of angles, into any equal given parts, I shall draw these few plain Corolaries.

1. *Corol.* Having in any right lined triangles, any one angle given, and the ratio of the other 2 angles to find these 2 angles, in degrees and minuts. e. g. let the given angle of a triangle be 110 deg. and the ratio of the other two angles be as 3 to 2. *Q.* What are the other 2 angles. *Ans.* Take the complement of 110 to 180, which is 70, and sect it into 5 equal parts by the method above delivered, $\frac{1}{5}$=14, this multiplied by 3 gives 42, and by 2 gives 28, wherefore the greater angle shall be 42, and the less 28. demonstrate 110+42+28 =180 and 3: 2: 42: 28.

2. *Cor.* From a Point given on a Line given, to erect an Isosceles triangle, one of whose sides produced shall come to a point given. (See the figure) let the point given be A, the line given AF, the limited point given below the line AF let the point be point D. the Praxis.

1. With radius AD describe the semicircle GFDE, so that G A shall cut AF at a right angle, and AE the like.

2. Draw the line AD, and cut the angle EAD into 3= parts.

3. Make the angle GAB=$\frac{1}{3}$ of EAD, and from B to D draw the line BD, cutting the line AF at C, the triangle ABC is an Isosceles, whose side BC is produced to D. The demonstration is evident from the trisection, and it hath singular uses in Architecture, and affording new Problems in *Geography* and *Navigation*, which I shall

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shall not at present inlarge upon, but leave to the ingenious Student to devise.

3. *Corol.* To measure any given angle in degrees and odd minuts, (if it have any) by a line of equal parts without a line of cords. Let the given angle be KAL (see the figure) with radius AL describe the quadrant LKL, and on that arch take the extent of the cord KL, and set it on the other side from L to a point on that arch.

Again, take the cord of that middle arch betwixt the point and K, and all these 3 cords join them together in a straight line, which make the cord of the quadrant (by a longer radius) as BC, which is easily done Geometrically.

Then because in each Quadrant there are 5400 minuts, divide the line BC by the Diagonal Scale into 5400 equal parts, and by the same Diagonal Scale measure on the line BC the length of the cord of KL, and whatever number of parts it giveth, that is the just quantity of that angle in minuts, which divided by 60 giveth the degrees, and if there be any Remainder, they are the odd minuts.

4. *Corol.* To project any angle without a line of cords, by a Diagonal Line of equal parts, whose quantity is given in degrees and minuts. Let the given angle to be projected, contain 36 deg. and 1 min. reduce the degrees into minuts, multiplying 36 by 60, the product is 2160, to which adding the odd minut, the sum is 2161.

Praxis.

Take off from the cord of the quadrant BC 2161 from C to D. on the point D erect the perpendicular ED, which extend to G, with the extent CD measure on the arch of the quadrant, from C to H.

Again, draw another arch with a less radius as AM (so that the radius be so long that the extent of DC reach not to EDG) and with that extent measure from M to I on that 2d arch.

4. From H to I draw a straight line, and produce it until it cut the perpendicular at K.

5. Draw the line AK, which shall give the angle required KAL=FAC. the demonstration whereof is apparent from the foregoing Problems.

If any discreet Persons desire further satisfaction or Information about these New Problems, if they please to call at my House, I shall be ready to endeavour to satisfie them; and whereas, to me the Method and Demonstration of these Matters, seem sufficiently evident, yet I have called the whole Praxis and Method here delivered, but an *Essay*, until it be found approved by able Artists, and have proposed it rather Problematically than Thetically, as is usual in such cases.

G. Keith.